CHARACTERIZATION OF BOUNDED DISTRIBUTIVE LATTICES IN A Γ – SEMIGROUP

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ABSTRACT: In this paper, we discussed to characterize some results on lattices of Γ – semigroup . Since a complemented element plays an important role in the study of lattices. So we give characterization of some results on lattice.

KEYWORDS: Γ – Semigroup, simple, additive and multiplicative Γ – idempotent Γ – semigroup, complemented elements in a Γ –semigroup, lattices.

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1. INTRODUCTION : In 1964, N. Nobusawa introduced the notion of a Γ – ring. W. E Barnes weakned slightly the conditions in the definition of Γ – ring in the sense of Nobusawa. Many fundamentals results in ring theory have been extended to Γ – ring by different authors obtaining various authors generalization analogous to corresponding parts in ring theory. In 1981, M. K sen and later in 1986 sen and saha in introduced the concept of Γ –semigroup as generalization of semigroup and ternary semigroup many classical notations of semigroup have extended to Γ –semigroup. From an algebraic point of view, semigroups provide the most natural common generalization of the groups and most of the techniques used in analysing semigroups are taken from ring theory, semi rings and group theory. In this paper, the efforts are made to characterize some results on lattice of Γ –semigroup. Furthermore, a completed element plays an important role in the study of lattices. So, we give characterization of some results on lattices in Γ –semigroup.

2. PRELIMINARIES:

DEFINITION 2. 1: Let S and Γ be two additive commutative semigroups. Then S is called a Γ – semigroup if there exist a mapping $S \times \Gamma \times S \to S$ denoted by $x\,\alpha\,y\ \forall x, y \in S\ \&\ \alpha \in \Gamma$ satisfying the following conditions.

(i) $x\,\alpha\,(y+z) = (x\,\alpha\,y) + (x\,\alpha\,z)$

(ii) $(y+z)\,\alpha\,x = (y\,\alpha\,x) + (z\,\alpha\,x)$

(iii) $x\,(\alpha+\beta)\,z = (x\,\alpha\,z) + (x\,\beta\,z)$

(iv) $x\,\alpha\,(y\,\beta\,z) = (x\,\alpha\,y)\,\beta\,z\ \forall x, y, z \in\ , \alpha, \beta \in \Gamma$

DEFINITION 2.2: A Γ –semigroup S is said to have a zero element if

$0\,\gamma\,x = 0\,x\gamma\,0$ *and* $x + 0 = x = 0 + x$ for *all* $x \in \mathbb{R}$ *and* $\gamma \in \Gamma$.

DEFINITION 2.3: A Γ –semigroup S is said to have identity element if $x\,\gamma\,1 = x = 1\,\gamma\,x$ for *all* $x, y \in S$ *and* $\gamma \in \Gamma$.

DEFINITION 2. 4: An element x of a Γ –semigroup S is said to be additive idempotent if and only if $x + x = x$. If every element of S is additive idempotent then S is called additive idempotent Γ –semigroup . It is denoted by $I^{+}(\Gamma(S))$.

DEFINITION 2. 5: An element x of a Γ –semigroup S is said to be multiplicative Γ – idempotent if there exists $\gamma \in \Gamma \ni x = x\,\gamma\,x$. If every element of S is multiplicative Γ – idempotent then S is called multiplicative Γ – idempotent Γ –semigroup. It is denoted by $I^{\times}(\Gamma(S))$.

DEFINITION 2. 6: A Γ – semigroup S is said to be Γ – idempotent if it is both additive idempotent and multiplicative Γ –idempotent.

NOTE 2.7: We will denote the set of all Γ – idempotent elements of a Γ – semigroup S by (Γ(S)) .

DEFINITION 2.8: A Γ –semigroup S with identity is simple if and only if $x + 1 = 1 = 1 + x\ \forall x \in \mathbb{R}$.

DEFINITION 2.9: A Γ –semigroup S is centreless if and only if $x + y = 0$ implies that $x = y = 0$.

DEFINITION 2. 10: The centre of a Γ –semigroup S is a subset of S consisting of all elements x of $\mathbb{R}$ $\ni x\, \gamma\, y = y\, \gamma\, x\ \forall y \in \mathbb{R}$ *and* $\gamma \in \Gamma$. It is denoted by $C(\mathbb{R})$.

DEFINITION 2. 11: Let x, y be elements of a Γ – semigroup S, then is Γ – interior y denoted by

$x \nabla y$ if *and only* if $\exists z \in \mathbb{R} \ni x\gamma z = z\gamma x = 0$ *and* $z + y = 1\ \forall \gamma \in \Gamma$.

DEFINITION 2.12: An element x is complemented if and only if $x \nabla x$. That is, $\exists y \in \mathbb{R} \ni x\gamma y = y\gamma x = 0$ *and* $x + y = 1\ \forall \gamma \in \Gamma$ .This element y of S is the complement of x is S.

We will denote complement of x *by* $x^{\perp}$.Clearly if $^{\perp}$ is complemented thenso is $x^{\perp}$ *and* $x^{\perp\perp} = x$.

LEMMA 2.13: Let S be a Γ -semigroup. Then

(i) is *simple* $\Leftrightarrow x = x + x\gamma y\ \forall x, y \in S\ \&\ \gamma \in \Gamma$ (ii)S is *simple* $\Leftrightarrow x = x + y\gamma x\ \forall x, y \in\in S\ \&\ \gamma \in \Gamma$

(iii)S is *simple* $\Leftrightarrow x\gamma y = x\gamma y + (x\beta z)y \forall x, y, z \in S\ \&\ \beta, \gamma \in \Gamma$

DEFINITION 2.14: Let a *and* b be two elements in partially ordered set $(A, \leq)$.An element 'c' is said to be an upper bound of a *and* b.If $a \leq c$ *and*

$b \leq c$, and an element 'c' is said to be a least upper bound of $a\ and\ b$ if c is an upper bound of $a\ and\ b$ and there is no other upper bound of $d\ as\ a\ and\ b\ such\ that\ d \leq c$. Similarly, an element c is said to be a greatest lower bound of $a\ and\ b$ if c is a lower bound of $a\ and\ b$ and if there is no other lower bound d of $a\ and\ b\ such\ that\ c \leq d$.

REMARK 2. 15: A lattice is a partially ordered set in which every two elements have a unique least upper bound and a unique greatest lower bound. Let $(A, \leq)$ be a lattice. We define an algebraic system $(A, \vee, \wedge)$ $where \vee\ and\ \wedge$ are two binary operations on A such that for $a\ and\ b$ in A, $a \vee b$ is equal to the g.l.b of $a\ and\ b$.

DEFINITION 2. 16: A lattice is said to be distributive lattice if the meet $(\wedge)$ operation distributes over the join$(\vee)$ operation and the join operation distributes over the meet operation. That is, for any $a, b\ and\ c$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)\ and$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

DEFINITION 2. 17 : An element a in a lattics $(A, \leq)$ is called a universal lower bound if for every element $b \in A$, we have $a \leq b$. An element in a lattice $(A, \leq)$ is called a universal upper bound if for every element $b \in A$, we have $b \leq a$.

We shall use '0' to denote the universal lower bound and '1' to denote the universal upper bound of a lattice (if such bounds exist)

DEFINITION 2. 18: Let $(A, \leq)$ be a lattice with universal lower bound and upper bounds '0 ' and '1' respectively. For an element a in A, an element b is said to be a complement of a if $a \vee b = 1\ and\ a \wedge b = 0$

DEFINITION 2.19: A lattice is said to be a complemented lattice if every element in the lattice has a complement.

NOTE 2.20: A complemented lattice must have universal lower and upper bounds.

DEFINITION 2.21: A complemented and distributive lattice is called Boolean lattice.

A Boolean lattice $(A, \leq)$ defines an algebraic system $(A, \vee, \wedge, \perp)$ is known as Boolean algebra, Where $\vee$, $\wedge$ *and* $\perp$ are the join, meet and the complementation operations respectively.

DEFINITION 2. 22: A Γ -semigroup S is lattice ordered if and only if it also has the structure of a lattice such that $\forall\, x, y \in S \text{ and } \gamma \in \Gamma$

(i) $x + y = x \vee y$

(ii) $x\,\gamma\, y = x \wedge y$ where partial order is one induced by the lattice structure on S.

THEOREM 2. 23: Let S be a Γ -semigroup. Then S is a bounded distributive lattice having unique minimal element 0 and unique maximal element 1 if and only if S is commutative, Γ -idempotent and simple Γ -semigroup.

Proof: Let S be a bounded distributive lattice having unique minimal element 0 and unique maximal element 1

Then S becomes a commutative, Γ -idempotent and simple Γ -semigroup by defining $x + y = x \vee y \text{ and } x\,\gamma\, y = x \wedge y\ \forall x, y \in \mathbb{R}$

Conversely, let S be a commutative, Γ -idempotent and simple Γ -semigroup. Then define a relation $\leq$ on S by $x \leq y$ if $x + y = y \text{ and } x\gamma y = x$

(i) $\leq x \text{ as } x + x = x\ \&\ x\gamma y = x$

(ii) $If\ x \leq y \text{ and } y \leq x \text{ then } x + y = y, y + x = x \text{ and } x\gamma y = x, y\gamma x = y. Thus\ x = y.$

(iii) If $x \leq y$ and $y \leq z$ then $x + y = y, y + z = z$ and $x\gamma y = x, y\gamma z = y$.

Thus $x + z = x + (y + z)$

$= (x + y) + z$

$= y + z = z$

And $x\gamma z = (x\gamma y)z$

$= x\gamma(y\gamma z)$

$= x\gamma y = x$

This implies that $x \leq z$

Hence $(S, \leq)$ is a partially ordered set. Define the operation $\vee$ and $\wedge$ on S by

$x + y = x \vee y$ and $x\gamma y = x \wedge y \ \forall x, y \in S, \gamma \in \Gamma$

Then it is easy to see that S is a bounded distributive lattice having unique minimal element 0 and unique maximal element 1.

3. CHARACTERIZATION OF BOUNDED DISTRIBUTIVE LATTICES.

PROPOSITION 3. 1: Let S be a Γ – semigroup. S is bounded distributive lattice having unique minimal element 0 and unique maximal element 1 if and only if it is commutative, Γ –idempotent, Γ –semiring and

$x \wedge (x \vee y) = x = x \vee (x \wedge y) \forall x, y \in S$

Proof: Let S be a bounded distributive lattice having unique minimal element 0 and unique maximal element 1.

Let $x, y, z \in S$

Since $x \vee (x \wedge y)$ is the join of x and $x \wedge y$

We have $x \leq x \vee (x \wedge y)$

Again, $x \leq x$ and $(x \wedge y) \leq x$. we have

$x \vee (x \wedge y) \leq x \vee x = x$

Therefore, $x \vee (x \wedge y) = x$

By principal of duality, $x \wedge (x \vee y) = x$.

Conversely, suppose that S be a commutative, Γ -idempotent Γ -semigroup and $x \wedge (x \vee y) = x = x \vee (x \wedge y) \forall x, y \in S$.

In theorem 2.23 it is sufficient to show that S is simple.

Putting $x = 1$ in $x \wedge (x \vee y) = x = x \vee (x \wedge y)$ as $x + y = x \vee$ $, x\gamma y = x \wedge y$.

We get $1(1 + y) = 1 \; \forall y \in \mathbb{R}$

That is $1 + y = 1 \; \forall y \in S$

Hence, S is simple.

THEOREM 3. 2: Let S be a Γ -semigroup. A commutative Γ-semigroup is a bounded distributive lattice if and only if it is simple multiplicative Γ -idempotent Γ -semigroup.

Proof: This is a direct proof of lemma 2.13. Proposition 3. 1 and theorem 2. 23

THEOREM 3.3: Let S be a simple Γ -semigroup then $(I^{\times}(\Gamma(S)), +)$ is a sub monoid of (S, +) and $(I^{\times}(\Gamma(S)) \cap C(S))$ is a bounded distributive lattice.

Proof: Let $x, y \in I^{\times}(\Gamma(S))$

Then $x\gamma x = x$ and $y\gamma y = y$

Therefore by lemma 2.13 we have

$$(x + y)\gamma(x + y) = (x + y)\gamma x + (x + y)\gamma y$$
$$= x\gamma x + y\gamma x + x\gamma y + y\gamma y$$
$$= x + y\gamma x + x\gamma y + y$$
$$= x + y.$$

So, $x + y \in I^{\times}(\Gamma(S))$ that is $I^{\times}(\Gamma(S))$ is closed under addition.

Since it contains 0. So $I^{\times}(\Gamma(S))$ is a sub monoid of (S, +).

Further, let $x, y \in (S) then\ x + y \in C(S)$

Therefore, $x + y \in I^{\times}(\Gamma(S)) \cap C(S)$.

Since $0, 1 \in I^{\times}(\Gamma(S)) \cap C(S)$.

So, $I^{\times}(\Gamma(S)) \cap C(S) \neq \emptyset$.

Let $x, y \in I^{\times}(\Gamma(S)) \cap C(S)$ then surely $x\gamma y \in I^{\times}(\Gamma(S)) \cap C(S)$

Thus $I^{\times}(\Gamma(S)) \cap C(S)$ is sub Γ -semigroup of S which is also simple,

since $x + 1 = 1\ \forall x \in I^{\times}(\Gamma(S)) \cap C(S)$.

Now the results follows from the theorem 3.2.

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